Overview: We want the probability that a randomly chosen number is not divisible by any square. We will show three things:

I.The desired probability is equal to the infinite product

$$P = (1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})(1 - \frac{1}{121})\dots$$

where the denominators are the *squares of primes*.

II. The *reciprocal* of that infinite product is equal to the infinite sum

$$1/P = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

where the denominators include *all squares* (both squares of primes and squares of composites).

III. The infinite sum in II above is also equal to $\pi^2/6$.

From these three statements, it follows that the desired probability is $6/\pi^2$. Statement I) is proved in paragraph 8. Statement II) is proved in paragraph 17. Statement III) is "proved" in paragraph 24; more precisely, it is made plausible via an argument that falls short of being an airtight proof. The remaining paragraphs give the airtight proof, which unfortunately yields less insight than the plausibility argument.

1. The problem: What is the probability that a randomly chosen number is not divisible by any square (other, of course, than 1)?

2. A simpler problem: What is the probability that our randomly chosen number is not divisible by 4? Well, let's see: 1, 2 and 3 are not divisible by 4; 4 is divisible by 4. 5,6 and 7 are not, but 8 is. 9, 10 and 11 are not, but 12 is. So 3/4 of all numbers are not divisible by 4; the probability that a randomly chosen integer is not divisible by 4 is 3/4, or 1-1/4.

3. Likewise, the probability that a randomly chosen number is not divisible by 9 is 8/9, or 1-1/9. The probability that a randomly chosen number is not divisible by 16 is 15/16, or 1-1/16. And so on.

4. Now what is the probability that a randomly chosen number is not divisible by 4 *and* not divisible by 9? To get the answer, multiply the probabilities for the two separate events: (3/4) times (8/9) is 2/3, and that's the answer.

5. Now what is the probability that a randomly chosen number is not divisible by 4, not divisible by 9, *and* not divisible by 16? You might be tempted to multiply $(3/4) \times (8/9) \times (15/16)$, but that's not right, because the events are not independent. In fact, as soon as you know a number is not divisible by 4, you *automatically* know it's not divisible by 16, so this question is equivalent to the question in paragraph 4. The answer is still $(3/4) \times (8/9) = 2/3$.

6. Now what is the probability that a randomly chosen number is not divisible by 4, not divisible by 9, not divisible by 16, and not divisible by 25? Answer: Multiply (3/4) (for not being divisible by 4) times (8/9) (for not being divisible by 9) times (24/25) (for not being divisible by 25). Skip the 15/16 (for not being divisible by 16) as explained in paragraph 5.

7. In paragraph 6, if you wanted to throw in the condition "not divisible by 36" , the answer wouldn't change, because "not divisible by 4" (or alternatively "not divisible by 9") already implies this.

8. In general, as you continue this process, you'll want to skip every square that is not the square of a *prime*. So the answer to the original question is: "Not divisible by any square means not divisible by 4, and not divisible by 9, and not divisible by 25, and not divisible by 49, and not divisible by 121, and ...; the probability of all this coming together is given by the infinite product

$$P = (1 - \frac{1}{4})(1 - \frac{1}{9})(1 - \frac{1}{25})(1 - \frac{1}{49})(1 - \frac{1}{121})\dots$$

9. Now let's look at the factors of P one by one. It will help to remember the identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

(If you don't remember this from freshman calculus, it's still easy to convince yourself it's true. Just multiply each side by 1-x. On the left you get 1. On the right, you get $(1-x)+(x-x^2)+(x^2-x^3)+\ldots$ which adds up (after much cancellation) to 1!

10. Let's apply paragraph 9 to the reciprocals of the factors in the number P (from paragraph 8). We get

$$\frac{1}{1-\frac{1}{4}} = 1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots$$

$$\frac{1}{1-\frac{1}{9}} = 1+\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\ldots$$

$$\frac{1}{1-\frac{1}{25}} = 1+\frac{1}{25}+\frac{1}{25^2}+\frac{1}{25^3}+\ldots$$

and so forth.

11. From the definition of P in paragraph 8 and the calculations in paragraph 9, we have

$$1/P = (1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\ldots)(1+\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\ldots)(1+\frac{1}{25}+\frac{1}{25^2}+\frac{1}{25^3}+\ldots)\cdots$$

12. How do you multiply out such a product? Let's remind ourselves how to handle a much simpler product, say

$$(A+B+C)(U+V+W)(X+Y+Z)$$

To expand this, first pick a term from the first sum (say A), then a term from the second (say W) and then a term from the third (say Y). Multiply to get AWY. Do this in every possible way and add the results:

$$\begin{gathered}AUX+AUY+AUX+AVX+AVY+AVZ+AWX+AWY+AWZ\\+BUX+BUY+BUX+BVX+BVY+BVZ+BWX+BWY+BWZ\\+CUX+CUY+CUX+CVX+CVY+CVZ+CWX+CWY+CWZ\end{gathered}$$

13. Applying the idea from paragraph 12 to the product in paragraph 11, choose a term from the first infinite sum (say 1), a term from the second (say $1/9^3$), a term from the third (say $1/25^2$), and terms from all the (infinitely many) sums that follow (say 1 in every case). Multiply these all together to get

$$1\times\frac{1}{9^3}\times\frac{1}{25^2}\times 1\times 1\times 1\times\cdots$$

or, in other words, $\frac{1}{9^3}\times\frac{1}{25^2}$, which is the same as $\frac{1}{455625}$, or, even more conveniently, $\frac{1}{675^2}$. That gives one term; now do the same thing, making the choices in every possible way, and add the results.

14. What terms show up when we do this? Here are some of them (in each case, everything after the $\cdots$ is assumed to be a 1):

$$1\times 1\times 1\times 1\times\cdots = 1$$

$$1\times\frac{1}{4}\times 1\times 1\times\cdots = \frac{1}{4}$$

$$1\times 1\times\frac{1}{9}\times 1\times\cdots = \frac{1}{9}$$

$$1\times\frac{1}{4^2}\times 1\times 1\times\cdots = \frac{1}{16}$$

$$1\times 1\times 1\times\frac{1}{5^2}\times\cdots = \frac{1}{25}$$

$$1\times\frac{1}{4}\times\frac{1}{9}\times 1\times\cdots = \frac{1}{36}$$

15. Now I want to ask: Exactly which numbers show up as denominators on the right side of the equations in Paragraph 14? You can see that we've got 4, 9, 16, 25, 36—or, in other words, 2^2, 3^2, 4^2, 5^2, 6^2. What about, say, 540^2? Yes, that's there, too, and here's why: First, observe that

$$540 = 2^2 \times 3^3 \times 5^1$$

Therefore

$$540^2 = 4^2 \times 9^3 \times 25^1$$

Therefore

$$1 \times \frac{1}{4^2} \times \frac{1}{9^3} \times \frac{1}{25} \times \cdots = \frac{1}{540^2}$$

16. The same trick we used in paragraph 15 demonstrates that *every square shows up as a denominator on the right side of the equations in paragraph 14.* The key observation is that *every number can be factored as a product of primes*, which is exactly what we needed to get started on proving that 540^2 occurs. (It's also important that every number can be factored as a product of primes in *exactly one way*, so that we know each square shows up as a denominator exactly once.) Likewise, 175 can be written as $5^2 \times 7$, so we have

$$1 \times 1 \times 1 \times \frac{1}{25^2} \times \frac{1}{49} \times \cdots = \frac{1}{175^2}$$

And likewise for every square.

17. Paragraph 11 gives $1/P$ as an infinite product of infinite sums. Paragraph 12 reminds us of how to expand such a product; it's an infinite sum of the terms that occur in Paragraph 14. Paragraph 16 tells us what those terms are: They are the reciprocals of (all!) the squares. Putting this together, we have

$$1/P = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

18. Now all we have to do is evaluate the infinite sum in Paragraph 17. I'll give away the answer: It's $\pi^2/6$. Therefore, P, which is the probability that we sought in Paragraph 1, is $6/\pi^2$, or approximately 60.79%.

19. Where does the $\pi^2/6$ come from? First I'll give you a plausibility argument for why you should expect that answer. Then I'll give a proof.

20. (Temporarily) forget everything we've done up till now and consider a completely new problem: How can we write the sine function as an infinite product? The sine function, remember, is a function that vanishes at $x = 0$, $x = \pi$, $x = -\pi$, $x = 2\pi$, $x = -2\pi$, etc. Therefore we should expect it to be a product of simple factors, one of which vanishes at 0, one of which vanishes at π, one of which vanishes at $-\pi$, one of which vanishes at 2π, etc. Like this for example:

$$\sin(x) = x(1 - \frac{x}{\pi})(1 + \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 + \frac{x}{2\pi})(1 - \frac{x}{3\pi})(1 + \frac{x}{3\pi})\cdots$$

21. Assuming our guess in paragraph 20 is right, we can combine terms to write

$$\sin(x) = x(1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2})\cdots$$

22. We can expand the infinite product in Paragraph 21 the same way we expanded the infinite product in earlier paragraphs: We pick one term from within each factor, multiply them together, and do this in every possible way. The terms we get will all involve powers of x. For example, there will be exactly one term that involves x to the first power, namely

$$x \times 1 \times 1 \times 1 \times 1 \cdots$$

23. What are all the ways to get a term with x to the third power? They include:

$$x \times \frac{-x^2}{\pi^2} \times 1 \times 1 \cdots = -\frac{x^3}{\pi^2}$$

$$x \times 1 \times \frac{-x^2}{4\pi^2} \times 1 \cdots = -\frac{x^3}{4\pi^2}$$

$$x \times 1 \times 1 \times \frac{-x^2}{9\pi^2} = -\frac{x^3}{9\pi^2}$$

And so forth. Each time, you "choose" x from the first factor (theres's actually no choice here!), a term involving x^2 from the factor of your choice, and all the rest 1's. The sum of the terms that occur in this way is

$$-\frac{x^3}{\pi^2} - \frac{x^3}{4\pi^2} - \frac{x^3}{9\pi^2} - \cdots$$

This is not the full expansion of the product in Paragraph 21; it's just the "x^3 part" of it.

23. On the other hand, remember from freshman calculus that we have the Taylor series

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

This tells you that the "x^3 part" of $\sin(x)$ is $-x^3/3! = -x^3/6$.

24. As long as you buy this talk of x^3 parts, we've computed the x^3 part of $\sin(x)$ in two different ways. Paragraph 22 gives

$$-\frac{x^3}{\pi^2} - \frac{x^3}{4\pi^2} - \frac{x^3}{9\pi^2} - \cdots$$

and paragraph 23 gives

$$-\frac{x^3}{6}$$

. Set these equal, divide each side by $-x^3$ and multiply each side by π^2, and you get

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \pi^2/6$$

which is exactly what I promised, back in Paragraph 18.

25. The argument above gives some insight into where the $\pi^2/6$ comes from. The argument to follow gives an actual proof.

26. Observe first that we can write

$$1 = \int_0^1 \int_0^1 1dxdy$$

$$1/4 = \int_0^1 \int_0^1 xydxdy$$

$$1/9 = \int_0^1 \int_0^1 (xy)^2dxdy$$

$$1/16 = \int_0^1 \int_0^1 (xy)^3dxdy$$

Et cetera.

27. Our goal is to add the reciprocals of all the squares. First let's add just the reciprocals of the *odd* squares, using the observtion in Paragraph 26:

$$1 + \frac{1}{9} + \frac{1}{25} + \cdots = \int_0^1 \int_0^1 [1 + (xy)^2 + (xy)^4 + \cdots]dxdy$$

28. Use the observation of paragraph 8, substituting $(xy)^2$ for x, to rewrite the integral on the right in Paragraph 27 as

$$\int_0^1 \int_0^1 \frac{1}{1-(xy)^2}dxdy$$

29. The integral in Paragraph 28 is tricky to evaluate, but here's the trick: Substitute

$$x = \frac{\sin(u)}{\cos(v)} \qquad y = \frac{\sin(v)}{\cos(u)}$$

so that

$$dudv = \frac{1}{1-(xy)^2}dxdy$$

and the limits of integration are transformed into the triangle with vertices at the origin, $(0, \pi/2)$, and $(\pi/2, 0)$. This transforms the integral to

$$\int_0^{\pi/2}\int_0^{(\pi/2)-v} 1dudv = \pi^2/8$$

29. Let E be the sum of the reciprocals of the even squares and O the sum of the reciprocals of the odd squares. Note that every even number is either twice an odd number or twice an even number, so every even square is either four times an odd square or four times an even square. It follows that $E = (1/4)(E+O)$, so that $(3/4)E = (1/4)O$.

30. Paragraph 28 computes that $O = \pi^2/8$. This and the equation at the end of Paragraph 29 give $E = \pi^2/24$. Finally, the sum of the reciprocals of all the squares is $E + O = \pi^2/24 + \pi^2/8 = \pi^2/6$. Tada!